



Mathematical Modeling of Regular and Irregular Shallow Water Waves Using Boussinesq Equation with Improved Dispersion

Prashant Kumar*, Vinita & Rupali

Department of Applied Sciences, National Institute of Technology Delhi, Delhi-110040, India

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A mathematical model has been formulated to analyze the behaviour of small amplitude linear and nonlinear shallow water waves in the coastal region. The coupled Boussinesq equations (BEs) are obtained from the Euler's equation in terms of velocities variable as the velocity measured from arbitrary distance from mean water level. BEs improves the dispersion characteristics and is applicable to variable water depth compared to conventional BEs, which is in terms of the depth-averaged velocity. The solution of the time-dependent BEs with kinematic and dynamic boundary conditions is obtained by utilizing Crank- Nicolson procedure of finite difference method (FDM). Further, the Von Neumann stability analysis for the Crank Nicolson scheme is also conducted for linearized BEs. The numerical simulation of regular and irregular waves propagating over the variable water depth is validated with the previous studies and experimental results. The present numerical model can be utilized to determine the wave characteristics in the nearshore region, including diffraction, refraction, shoaling, reflection, and nonlinear wave interactions. Therefore, the current model provides a competent tool for simulating the water waves in harbour or coastal regions for practical application.

Keywords: Boussinesq Equations, Shallow Water Waves, Finite Difference Method, Crank Nicolson Discretization,

1 Introduction

An important objective in harbour and coastal engineering is concerned with the reliable prediction of the wave characteristics such as shoaling, refraction, diffraction and harmonic interaction of wave near coastal boundaries and offshore structures. The depth averaged Boussinesq model is a nonlinear shallow water model which, include the nonlinear phenomenon. This eliminates the vertical dependency of the water wave velocity variable from the BEs. Thus, satisfactory modelling of all these physical movements present through-out the coastal regions, from offshore region to near shorelines, is significantly important for practical applications. BEs has been first introduced by Peregrine for water of variable bathymetry to determine the nonlinear transformation of multidirectional surface wave¹. Later on, numbers of investigators have made efforts to revise the standard BEs and derive the extended/alternative form for the BEs²⁻⁵. The dispersion properties of various forms are improved by adding additional dispersion terms

in the BEs^{2,3}. Numerical methods have been developed by investigators to study the water wave agitation and hydrodynamics in swash and surf zones⁶⁻¹². Several mathematical models are developed incorporating the effect of refraction, diffraction, constant and variable bathymetry^{13,14}. Boussinesq equations (BEs) are better described the accurate analysis of the nonlinear wave transformation with physical effects such as wave breaking, shoaling, harmonic interaction^{8,15-17}. BEs significantly models the practical problems generating from wave disturbances in port, harbour and coastal areas^{18,19}.

In recent literature, various numerical approaches such as FDM, finite element analysis (FEA) have been proposed to obtain solution of BEs. The model based on FDM was adopted by many researchers to obtain the solution of improved or extended BEs^{5,18,20,21}. As the FDM was not applicable to complex geometry, researchers focus more on unstructured FEMs for approximating the extended BEs^{8,22,23}. A hybrid numerical approach based on FDM and finite volume method (FVM) is established to analyze the wave transformation, run up, and breaking in

Corresponding author (Email: prashantkumar@nitdelhi.ac.in)

real complex coastal domain^{8,23,24}. Recently, a nonlinear Boussinesq mode using hybrid FDM and FVM to predict the breaking of regular wave are presented by Tatlock²⁵.

In this paper, BEs is described in terms of velocity variables and water wave surface elevation, which has been evaluated by governing equations of motion. Boussinesq equations are improved the dispersion characteristics of waves. The FDM is used to obtain the solution of resulting time-dependent BEs in one variable with the Crank-Nicolson iterative. Error and convergence analysis is implemented to investigate the numerical accuracy of present scheme. The numerical results of multi waves (irregular and regular waves) are validated with the existed experimental data. An efficient tool to analyze the transformation of waves through deep to shallow water is discussed in the present study.

2 Mathematical formulation

Consider fluid as inviscid, non-compressible, and fluid flow as non-rotational. Consider the two-dimensional wave field with water surface elevation $\xi(x, y, t)$ at time t propagating over a variable depth $h(x, t)$. The Cartesian co-ordinate system (x, z) is assumed to have x -axis with the horizontal direction and z -axis with upward negative. The mathematical model is developed by using the incompressible Euler's equations of motion and continuity equation. The non-dimensional system variables represented by symbols x, y, z and t are expressed as follows:

$$x = \frac{X}{L}, \quad y = \frac{Y}{L}, \quad z = \frac{Z}{h_0}, \quad t = \frac{\sqrt{gh_0}}{L} T \quad \dots (1)$$

$$u = \frac{h_0}{A\sqrt{gh_0}} U, \quad v = \frac{h_0}{A\sqrt{gh_0}} V, \quad w = \frac{h_0^2}{AL\sqrt{gh_0}} W, \quad \dots (2)$$

$$h = \frac{H}{h_0}, \quad \xi = \frac{\xi^*}{A}, \quad p = \frac{P}{\rho g A}$$

Where P is dimensional form of pressure. Here $\xi^*(X, Y, T)$ is refer to surface elevation in the dimensional form and A represents wave amplitude.

The non-dimensional form of continuity and Euler equations are defined as:

$$\varepsilon^2 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{\partial w}{\partial z} = 0 \quad \dots (3)$$

$$\varepsilon^2 \frac{\partial u}{\partial t} + \varepsilon^2 \psi \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) + \psi w \frac{\partial u}{\partial z} + \varepsilon^2 \frac{\partial p}{\partial x} = 0 \quad \dots (4)$$

$$\varepsilon^2 \frac{\partial v}{\partial t} + \varepsilon^2 \psi \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) + \psi w \frac{\partial v}{\partial z} + \varepsilon^2 \frac{\partial p}{\partial y} = 0 \quad \dots (5)$$

$$\psi \frac{\partial w}{\partial t} + \varepsilon^2 \left(u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} \right) + \frac{\psi^2}{\varepsilon^2} w \frac{\partial w}{\partial z} + \psi \frac{\partial p}{\partial z} + 1 = 0 \quad \dots (6)$$

The parameter $\psi = A/h_0$ is a measure of non-linearity and $\varepsilon = h_0/L$ measures frequency dispersion. The non-rotational condition is expressed as $\nabla \times (u, v, w) = 0$. The boundary conditions for the fluid motion are

1 The dynamic boundary condition (on free surface) given as:

$$p = p_{atm} \quad \text{at} \quad z = \psi \xi(x, y, t) \quad \dots (7)$$

2 The kinematic boundary condition (on free surface) is written as:

$$w = \varepsilon^2 \left(\frac{\partial \xi}{\partial t} + \psi u \frac{\partial \xi}{\partial x} + \psi v \frac{\partial \xi}{\partial y} \right) \quad \text{at} \quad z = \psi \xi(x, y, t) \quad \dots (8)$$

4 The rigid bottom condition is

$$w = -\varepsilon^2 \left(u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} \right) \quad \text{at} \quad z = -h(x, y) \quad \dots (9)$$

where p_{atm} refers to the atmospheric pressure. On integrating Eq. (3)-(6) with using boundary conditions $-h$ to $\psi \xi$ and it's moving in the same way as in Nwogu⁴. The following form of BEs are obtained as,

$$\frac{\partial \xi}{\partial t} + \nabla \cdot \left[(h + \psi \xi) \mathbf{u}_\beta \right] + \varepsilon^2 \nabla \cdot \left(\frac{z_\beta^2}{2} - \frac{h^2}{6} \right) h \nabla (\nabla \cdot \mathbf{u}_\beta) \quad \dots (10)$$

$$+ \varepsilon^2 \nabla \cdot \left(z_\beta + \frac{h}{2} \right) h \nabla (\nabla \cdot (h \mathbf{u}_\beta)) = 0$$

$$\frac{\partial \mathbf{u}_\beta}{\partial t} + \nabla \xi + \psi (\mathbf{u}_\beta \cdot \nabla) \mathbf{u}_\beta + \varepsilon^2 \frac{z_\beta^2}{2} \nabla \left(\nabla \cdot \frac{\partial \mathbf{u}_\beta}{\partial t} \right) + \varepsilon^2 z_\beta \nabla \left(\nabla \cdot \left(h \frac{\partial \mathbf{u}_\beta}{\partial t} \right) \right) = 0 \quad \dots (11)$$

$$\frac{\partial u}{\partial t} + g \frac{\partial \xi}{\partial x} + h^2 \beta \frac{\partial^3 u}{\partial x^2 \partial t} = 0 \quad \dots (15)$$

The periodic wave for small amplitude, wave number k , and frequency ω is given as:

$$\xi(x, t) = a_0 \exp(i(kx - \omega t)), \quad \dots (16)$$

$$u(x, t) = u_0 \exp(i(kx - \omega t)),$$

These are the BEs which are utilized to analyze the irregular multi-directional waves propagating horizontally over the water of varying depth. Eq. 10 and Eq. 11 are combined to obtain non-dimensional equation in one variable for the shallow water wave propagation.

$$\frac{\partial^2 \xi}{\partial t^2} - \frac{\partial^2 \xi}{\partial x^2} - \psi \frac{\partial^2}{\partial t^2} (\xi^2) - \frac{\psi}{2} \frac{\partial^2}{\partial x^2} (\xi^2) + \varepsilon^2 \left(2\beta + \frac{1}{3} \right) \frac{\partial^4 \xi}{\partial x^4} = 0 \quad \dots (12)$$

The BEs are described the applicability for frequency dispersivewaves in shallow water and intermediate water. The dimensional form of nonlinear 1-D BEs is written as:

$$\frac{\partial^2 \xi^*}{\partial t^2} - C^2 \frac{\partial^2 \xi^*}{\partial x^2} - \frac{\partial^2}{\partial t^2} (\xi^{*2}) - \frac{C^2}{2h} \frac{\partial^2}{\partial x^2} (\xi^{*2}) + C^2 h^2 \left(2\beta + \frac{1}{3} \right) \frac{\partial^4 \xi^*}{\partial x^4} = 0 \quad \dots (13)$$

Where $C = \sqrt{gh}$ which is define the phase speed for the shallow water and $\beta = \left(\frac{z_\beta}{h} \right)^2 / 2 + \frac{z_\beta}{h}$.

3 Dispersion characteristics

The comparison of linear dispersion relation based on the present form for BEs and Airy wave theory dispersion relation of the BEs for different water depth is discussed. The Eq. 10 and Eq. 11 have been transformed in linearized form as given and for horizontal dimension and for constant bathymetry is expressed as:

$$\frac{\partial \xi}{\partial t} + h \frac{\partial u}{\partial x} + h^3 \left(\beta + \frac{1}{3} \right) \frac{\partial^3 u}{\partial x^3} = 0 \quad \dots (14)$$

where a_0 and u_0 denotes wave amplitude and velocity amplitude, respectively. Substituting above Eq. 16 in Eq. 14 and Eq. 15 and letting discriminant as zero to obtain the dispersion relation's nontrivial solution, the following relation is written as,

$$C_B^2 = \frac{\omega^2}{k^2} = gh \left[\frac{1 - \left(\beta + \frac{1}{3} \right) (kh)^2}{1 - \beta (kh)^2} \right] \quad \dots (17)$$

Where C_B is phase speed, $k = 2\pi/L$ denotes the wave number. The linear dispersion relation from Airy wave theory is given as:

$$C_{Airy}^2 = \frac{\omega^2}{k^2} = gh \left[\frac{\tanh(kh)}{kh} \right] \quad \dots (18)$$

The concept of group velocity associated with wave energy is significant for the study of wave propagation and is related to phase velocity as follows

$$C_{gB} = \frac{\partial \omega}{\partial k} = C \left(1 - \frac{\left(\frac{(kh)^2}{3} \right)}{\left[1 - \beta (kh)^2 \right] \left[1 - \left(\beta + \frac{1}{3} \right) (kh)^2 \right]} \right) \quad \dots (19)$$

A comparison of normalized form of phase speed with airy wave theory is displayed in left panel of Fig. 1 corresponding to different values of β . The abscissa is described as the relative water depth, which is equal to ratio of water depth h to deepwater wavelength $L = 2\pi g/\omega^2$. The deepwater limit depth corresponding to relative depth *i.e.*, h/L is 0.5. The parameter β in BEs enhances the linear dispersion characteristics of system. The error of phase speed for shallow water depth is defined by the optimum depth which is less than 2%. The normalized form of group velocities w.r.t. airy wave theory are displayed in right panel of the Fig. 1 for different values of β , in which abscissa represents relative water depth function. For the intermediate relative water depth less than 0.3, the difference between the behaviour of group velocities and phase speeds for the Boussinesq model, and airy wave theory are quite similar.

4 Numerical solution of BEs

The finite difference Crank-Nicolson approximation are defined for time t variable of partial derivatives and the central difference scheme approximates the derivatives corresponding to the space variable. The space step size is Δx and Δt refer to the time step size. The solution procedure contains the three steps. At time step $t = n\Delta t$, value of variable $t = \left(n + \frac{1}{2}\right)\Delta t$ is predicted from given values at $t = n\Delta t$. Then approximated values at time $t = \left(n + \frac{1}{2}\right)\Delta t$ are utilized in corrector stage to determine values at time $t = (n+1)\Delta t$. The value of ξ

at the grid point $(x, t) = (i\Delta x, n\Delta t)$ is denoted by ξ_i^n , where $i=1,2,3,\dots,N+1$, and $n=1,2,3,\dots$. Consider the computational domain is $[x_0, x_l]$ with grid size Δx such that $x_i = i\Delta x$ where $i = 0, 1, 2, \dots, N$. Here $n = 1, 2, 3, \dots, N-1$ are the interior points of the domain, and $i = 0$ and $i = N$ are boundary points. For small-amplitude regular wave, velocity components are obtained by using continuity equation and surface elevation. Incident boundary condition at $x = 0$ is defined as:

$$\xi(0, t) = \left(\frac{kh_0}{\omega}\right) \left(1 - \left(\beta + \frac{1}{3}\right)(kh)^2\right) (u(0, t)) \dots (20)$$

The 2nd order partial derivative for the velocity $\xi_{xx}(0, t)$ is also specified at the incident boundary. The perfectly absorbing boundary condition is

$$\frac{\partial \xi}{\partial t} + C_0 \frac{\partial \xi}{\partial x} = 0 \dots (21)$$

The outgoing absorbing boundary is defined as in the form of first-order partial derivative of surface elevation w.r.t. x space.

$$\frac{\partial \xi}{\partial x} \Big|_{i=N,n} = \frac{3\xi_N^n - 4\xi_{N-1}^n + \xi_{N-2}^n}{2\Delta x} \dots (22)$$

5 Discretization form of BEs

An iterative Crank-Nicolson scheme for discretizing the nonlinear and dispersive BEs is used. Assuming $F = \xi^2$, the following approximation for the BEs defines as:

$$\left(\frac{\partial^2 \xi}{\partial t^2}\right)^{(n+1)} - C^2 \left(\frac{\partial^2 \xi}{\partial x^2}\right)^{(n+1)} - \left(\frac{\partial^2 F}{\partial t^2}\right)^{(n+1)} + C^2 h^2 \left(2\beta + \frac{1}{3}\right) \left(\frac{\partial^4 \xi}{\partial x^4}\right)^{(n+1)} - \frac{C^2}{2h} \left(\frac{\partial^2 \xi}{\partial x^2}\right)^n = 0 \dots (23)$$

The discretized form of nonlinear BEs in Eq. 23 is defined and resulting system equation is expressed as:

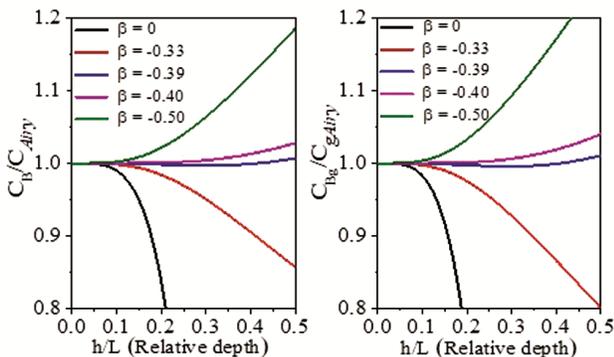


Fig. 1 — The comparison between of normalized form of phase speeds and group velocities for different values of $\beta = \{0, -0.33, -0.39, -0.40, -0.50\}$ are displayed in left panel, and right panel.

$$\begin{aligned}
 & r_1 \xi_{i+2}^{n+1} - (r + 4r_1) \xi_{i+1}^{n+1} + (1 - 2r + r_1) \xi_i^{n+1} \\
 & - (r + 4r_1) \xi_{i-1}^{n+1} - r_1 \xi_{i-2}^{n+1} = S_i \\
 & \dots
 \end{aligned} \tag{24}$$

The right hand side of Eq. 24 is S_i , which is given as follows,

$$\begin{aligned}
 S_i = & -r_1 \xi_{i+2}^n + (r + 4r_1) \xi_{i+1}^n + (2 - 2r - 6r_1) \xi_i^n + \\
 & (r + 4r_1) \xi_{i-1}^n + r_1 \xi_{i-2}^n \\
 & - \xi_i^{n-1} + r_2 (F_{i+1}^n - 2F_i^n + F_{i-1}^n) + \\
 & (F_i^{n+1} - 2F_i^n + F_i^{n-1})
 \end{aligned} \tag{25}$$

Where $i=1,2,3\dots N-1$, and $n=1,2,3,\dots \in Z^+$ and the coefficients values are

$$\begin{aligned}
 r &= \frac{C^2 \Delta t^2}{2 \Delta x^2}, \quad r_1 = \frac{C^2 h^2 \Delta t^2}{2 \Delta x^4} \left(2\beta + \frac{1}{3} \right), \text{ and} \\
 r_2 &= \frac{1}{2h} \left(\frac{C^2 \Delta t^2}{\Delta x^2} \right),
 \end{aligned} \tag{26}$$

The matrix form of the system of equation in Eq. 73 is written as

$$\begin{bmatrix}
 r_1 & -(r+4r_1) & (1+2r+6r_1) & -(r+4r_1) & r_1 & 0 & 0 \\
 0 & r_1 & -(r+4r_1) & (1+2r+6r_1) & -(r+4r_1) & -r_1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 0 & 0 & 0 & 0 & 0 & r_1 & -(r+4r_1)
 \end{bmatrix}
 \begin{bmatrix}
 \xi_0^{n+1} \\
 \xi_1^{n+1} \\
 \xi_2^{n+1} \\
 \vdots \\
 \vdots \\
 \xi_N^{n+1}
 \end{bmatrix}
 =
 \begin{bmatrix}
 S_0 \\
 S_1 \\
 S_2 \\
 \vdots \\
 \vdots \\
 S_N
 \end{bmatrix}
 \tag{27}$$

The above discretization Eq. 27 simplifies to give the following iterative form $R \xi_i^{n+1} = S_i$. The numerical scheme is applied to all internal grid points and similar to be performed for the boundary nodes. This numerical solution involves the implicit form of TDMA for ξ . The discretization form of BEs is represented in term of a penta-diagonal system of equations. At each iteration, the Gauss elimination with the partial pivoting scheme is employ to achieve the solution of system.

6 Result

The computational domain has been determined the surface elevation with respect to space for the regular periodic wave model with constant water depth 0.10 m near the shore with slope 1:25. The computational domain is discretized with the grid step size in space $\Delta x = 0.020$ m and grid size in time $\Delta t = 0.025$ sec. The wavelength of the water waves is 1.127 m. The wave period of the numerical model $T = 1$ sec, and height is $H = 0.06$ m.

Figure 2 shows that the surface elevation w.r.t. different values of $\beta = \{-0.33, -0.39\}$ is representing the different wave groups and phase speed as the water depth decreases. It is observed that some interaction between the frequencies occurs in the numerical model, as wave height at the intermediate water depth gives well match. In Fig. 3, the surface

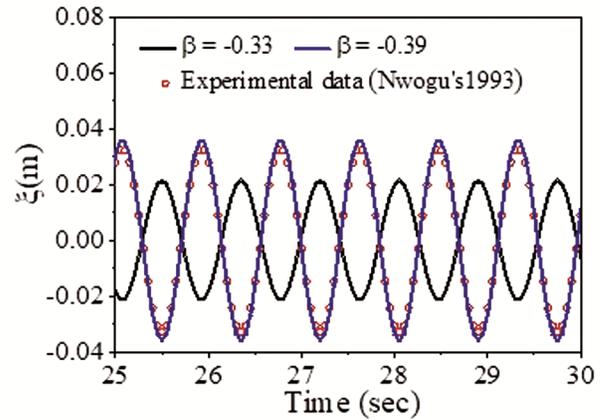


Fig. 2 — Validation of the surface elevation of regular waves with experimental data at depth $h=0.24$ m for different value of $\beta = \{-0.33, -0.39\}$ with wave period $T = 1$ sec.

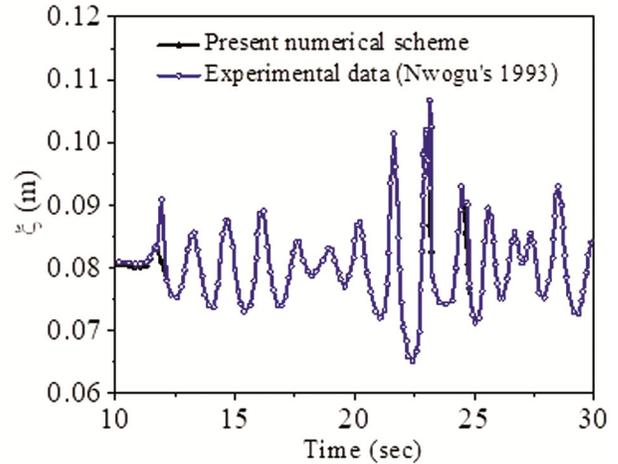


Fig. 3 — Comparison of surface wave elevation of present numerical scheme with experimental data for irregular wave shoaling with $\beta = -0.39$ with wave period $T = 1$ sec.

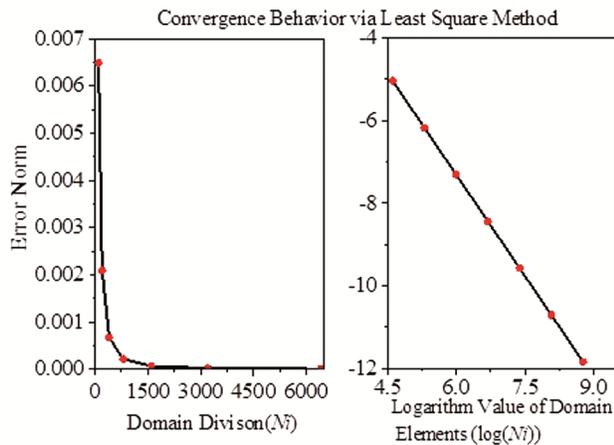


Fig. 4 — The left panel displays the error plot with respect to discrete domain divisions, and the right panel is the logarithmic error plotted for the logarithmic value of different domain divisions.

elevation with irregular waves is evaluated with time series of 10 sec-30 sec at the inflow boundary, water depth is $h = 0.56$ m with wave period $T = 0.35$ second free parameter related to water depth is $\beta = -0.39$. The shoaling effects of irregular waves in deep to intermediate and shallow water have been determined from the present model and it is also substantially changed during the shoaling process. The Numerical results are obtained from the present numerical scheme for the irregular waves whose has been matched with experimental data from Nwogu⁴.

The current numerical model is discussed the convergence analysis. The numerical scheme approaches the true solution of problem as mesh size approaches close to 0. The length of domain is $L = 1.127$ m, and depth is $h = 0.56$ m. Assuming the domain is discretized into uniform finite segments with $N_1 = 100$, $N_2 = 2 \times N_1$ and, $N_3 = 2 \times N_2$ and so on. The order of convergence is calculated by Least square method, and the iterative method is utilized to analyze the error norm approximation. In Fig. 4, the left panel shows the error plot computed for different domain divisions, and the right panel represents the logarithmic value of the error plot. The order of the convergence for the numerical model is 1.64. The present numerical scheme gives better match of the error norm approximation with Nwogu's experimental data.

7 Conclusion and discussion

Boussinesq equations in the form of surface elevation have been derived to examine the dispersion

properties such as shoaling, refraction, and diffraction of linear and nonlinear waves transformation of water waves. The BEs has been improved the dispersion characteristics properties in intermediate and shallow water depth. Forward time central space (FTCS) with Crank Nicolson Scheme is utilized for the numerical solution of BE. The present numerical model has used to analyze the variations in wave elevation for the multi-waves with respect to different time intervals and the timeanalysis of the surface elevation is performed for irregular and regular waves with different depth. The convergence analysis of the BEs is defined for shallow water wave propagation which is evaluated the convergence analysis 1.64. The good agreement of the Boussinesq model results has been showed the capability of simulating shallow water wave propagation for realistic coastal region.

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